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Partial monotonizations of Hamiltonian cycle polytopes: dimensions and diameters

Gerard Sierksma[a], Ruud H. Teunter[b,*,1]

[a] University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands

[b] Department of Mechanical Engineering, Industrial Management Division, Aristoteles University of Thessaloniki, P.O. Box 461, 54006 Thessaloniki, Greece

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Abstract

In this paper we study partial monotonizations and level polytopes of the Hamiltonian Cycle Polytope, also called the symmetric Traveling Salesman Polytope. The kth Level Polytope is the convex hull of the characteristic vectors corresponding to sets of k edges in K_n that can be extended to Hamiltonian cycles ($n \geqslant 3$). For $0 \leqslant \alpha \leqslant k$, the α-monotonization of the kth Level Polytope is the convex hull of the characteristic vectors corresponding to sets of at least α and at most k edges in K_n that can be extended to Hamiltonian cycles ($n \geqslant 3$). It is shown that for $\alpha < k$, α-monotonizations of level polytopes are full dimensional. We give upper and lower bounds for the diameters of the α-monotonizations and determine the number of 0-faces of the level polytopes and α-monotonizations. The main result of this paper is a proof that the diameter of the monotone Hamiltonian Cycle Polytope and the monotone Hamiltonian Path Polytope are each $\theta(\log n)$.

Keywords: Hamiltonian cycle polytope; Traveling salesman polytope; Monotonization

1. Introduction

The usual lower monotonizations extend a given polytope 'all the way down' to the origin, creating a new full-dimensional polytope with the original polytope as a face. However, the number of vertices may then increase drastically. One may ask: are there smaller (partial) monotonizations that are meaningful and full dimensional? In this paper, we consider the Hamiltonian Cycle Polytope, which is the well-known feasible region in the polyhedral LP-formulation of the Traveling Salesman Problem,

* Corresponding author. Tel.: +30-31-99-60-23; fax: +30-31-99-60-18

E-mail addresses: g.sierksma@eco.rug.nl (G. Sierksma), teunter@yahoo.com (R.H. Teunter).

[1] The author was working at the University of Groningen when the first version of this paper was written.

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and we define and study dimensions and diameters of its partial monotonizations. Let $K_n = (V, E)$ be the complete graph on n nodes, $n \geqslant 3$. K_n has $m = |E| = \frac{1}{2}(n)(n-1)$ edges and contains $\frac{1}{2}(n-1)!$ Hamiltonian cycles. By $\mathscr{A}_n(k)$ is denoted the subset of the power set of E consisting of all sets of k edges in K_n that can be extended to Hamiltonian cycles. For any subset S of E, x^S denotes the characteristic vector of S; moreover, $E(x)$ is the subset of E for which x is the characteristic vector. The polytope $Q_n(k) = \operatorname{conv}\{x^S \in \mathbb{R}^m \mid S \in \mathscr{A}_n(k)\}$ is called the kth Level Polytope. Hence, $Q_n(n)$ is the well-known Hamiltonian Cycle Polytope, also known as the Traveling Salesman Polytope (see e.g. [4]), and $Q_n(n-1)$ is the Hamiltonian Path Polytope. Partial monotonizations of level polytopes are extensions of it towards the origin in the following way. For $k = 1, \dots, n$ and $\alpha = 0, \dots, k$, the α-monotonization of $Q_n(k)$, is denoted and defined as $\tilde{Q}_n(\alpha, k) = \operatorname{conv}\{x^S \in \mathbb{R}^m \mid S \in \mathscr{A}_n(\alpha) \cup \cdots \cup \mathscr{A}_n(k)\}$. The 0-monotonization $\tilde{Q}_n(0, n)$ of Q_n, is the well-known Monotone Traveling Salesman Polytope; see [1]. The *skeleton* of a polytope P, denoted by skel(P), is the graph with vertex set (notation: vert(P)) the 0-faces of P and with edge set the 1-faces of P. The *diameter* of a graph G, denoted by diam(G), is the maximum distance between any pair of vertices, where the distance between a pair of vertices is defined as the smallest number of edges forming a path between the two vertices.

2. Dimensions of level polytopes and partial monotonizations

One of the reasons of studying monotonizations is the fact that they are full dimensional. In [4] it is shown that the Hamiltonian Cycle Polytope is not full dimensional, while its 0-monotonization is. One may ask what is the smallest full-dimensional partial monotonization of the Hamiltonian Cycle Polytope? In Theorem 2.1, we determine the dimensions of level polytopes and partial monotonizations, and show that level polytopes are faces of partial monotonizations 'beginning' or 'ending' at that level.

Theorem 2.1. *Let $1 \leqslant k \leqslant n$, $0 \leqslant \alpha \leqslant k$ and $m = \frac{1}{2}(n)(n-1)$. Then the following holds.*
(a) $\dim(Q_n(0)) = 0$.
(b) $\dim(Q_n(k)) = m - 1$ *for* $k = 1, \dots, n-1$.
(c) $\dim(Q_n(n)) = m - n$.
(d) $\dim(\tilde{Q}_n(\alpha, k)) = m$ *for* $\alpha < k$.

Proof. (a) Trivial, since $Q_n(0) = \{0\}$. (b) Since all $u \in Q_n(k)$ satisfy the equation $\sum_e u_e = k$ it follows that $\dim(Q_n(k)) \leqslant m - 1$. On the other hand, let $\lambda u = \lambda_0$ be a linear equation satisfied by all $u \in Q_n(k)$. For any two edges e, $f \in E(K_n)$ there exists a Hamiltonian cycle containing e and f. Therefore (since $k \leqslant n-1$), there exists a path $P \in \mathscr{A}_n(k-1)$ such that both $P + e$ and $P + f$ are in $\mathscr{A}_n(k)$. Then $\lambda x^{P+e} = \lambda x^{P+f}$, implying $\lambda_e = \lambda_f$ for all $e, f \in E(K_n)$. Hence, all λ_e have the same value and the equation $\lambda x = \lambda_0$ must be a multiple of $\sum_e u_e = k$. This proves that $\dim(Q_n(k)) = m - 1$. (c) See [4]. (d) Direct consequence of part (b). □

As a consequence of Theorem 2.1(d), we have that the Hamiltonian Path-Cycle Polytope $\mathrm{conv}\{x^S \in \mathbb{R}^m \mid S \in \mathscr{A}_n(n-1) \cup \mathscr{A}_n(n)\}$ is the smallest full-dimensional partial monotonization of the Hamiltonian Cycle Polytope.

Theorem 2.2. *Let $0 \leqslant k \leqslant n$, $0 \leqslant l \leqslant n$ and $0 \leqslant \alpha \leqslant k$. Then the following holds.*

$Q_n(l)$ is a face of $\tilde{Q}_n(\alpha,k)$ iff $l \in \{\alpha,k\}$. Moreover,
$Q_n(l)$ is a facet of $\tilde{Q}_n(\alpha,k)$ if $l \in \{\alpha,k\}$, unless $l=\alpha=0$ or $l=k=n$.

Proof. We only proof the first part. The second part follows directly from the first part and Theorem 2.1.

$Q_n(\alpha)$ (resp. $Q_n(k)$) is the set of all minimizers (resp. maximizers) of $\sum_e u_e = k$ on $\tilde{Q}_n(\alpha,k)$, hence both are faces of $\tilde{Q}_n(\alpha,k)$. Now, if $\alpha < l < k$ then it suffices to construct a point $w \in Q_n(l)$ such that $w \in \mathrm{conv}(u,v)$ for some $u,v \in \tilde{Q}_n(\alpha,k) \setminus Q_n(l)$. For this, choose $u \in Q_n(l+1)$ and (since $l+1 \geqslant 2$) two edges $f,g \in E(u)$. Define $v = u - x^f - x^g$ and $w = (1/2)u + (1/2)v$. Letting $u^f = u - x^f \in Q_n(l)$ and $u^g = u - x^g \in Q_n(l)$, we have $w = (1/2)u^f + (1/2)u^g \in Q_n(l)$, a contradiction. □

3. Diameters of partial monotonizations

In this section, we give lower and upper bounds for the diameters of monotonizations.

Theorem 3.1. *The diameter of the skeleton of $\tilde{Q}_n(0,n-1)$ is at most $4\lceil {}^{(5/4)}\log(n-1)\rceil + 4$, and the diameter of the skeleton of $\tilde{Q}_n(0,n)$ is at most $4\lceil {}^{(5/4)}\log n\rceil + 4$.*

Proof. We first prove that the diameter of $\tilde{Q}_n(0,n-1)$ is at most $4\lceil {}^{(5/4)}\log(n-1)\rceil + 4$. Let $u,v \in \mathrm{vert}(\tilde{Q}_n(0,n))$. Any pair of edges e_1 and e_2, with $e_1 \in E(u) \setminus E(v)$ and $e_2 \in E(v) \setminus E(u)$, are called noncombinable with respect to (NC w.r.t) u and v iff $(E(u) \cap E(v)) \cup \{e_1,e_2\} \notin \mathscr{A}_n(0) \cup \cdots \cup \mathscr{A}_n(n)$. Note that if e_1 and e_2 are NC w.r.t. u and v, then the addition of e_1 and e_2 to $E(u) \cap E(v)$ causes either a subcycle or a claw (a subgraph of three edges with a common node). It is well known that for two nonadjacent vertices u and v of $\tilde{Q}_n(0,n)$ there is a strict convex combination of vertices in $\mathrm{vert}(\tilde{Q}_n(0,n)) \setminus \{u,v\}$, say $x = \sum_{i=1}^T \lambda_i y_i$ with $\sum_{i=1}^T \lambda_i = 1$, $\lambda_1,\ldots,\lambda_T > 0$ and $\{y_1,\ldots,y_T\} \in \mathrm{vert}(\tilde{Q}_n(0,n)) \setminus \{u,v\}$, such that $x \in \mathrm{conv}(u,v)$; see [5]. Such a set $\{y_1,\ldots,y_T\}$ is called a nonadjacency-certificate for u,v.

Claim 3.1. *Let u and v be nonadjacent vertices of $\tilde{Q}_n(0,n)$ and let e_1,e_2 be a pair of NC edges w.r.t. u and v, with $e_1 \in E(u) \setminus E(v)$ and $e_2 \in E(v) \setminus E(u)$. Let $\{y_1,\ldots,y_T\}$ be a nonadjacency-certificate for u,v. Then for each $k=1,\ldots,T$ it holds that either $e_1 \in E(y_k)$ or $e_2 \in E(y_k)$.*

A proof of this claim is given in [10].

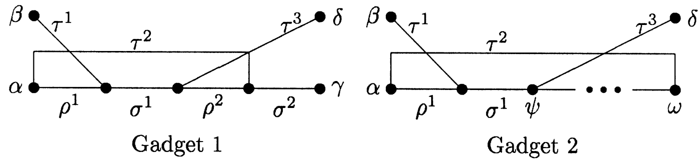

Fig. 1. Two gadgets.

For $u \in \mathrm{vert}(\tilde{Q}_n(\alpha, k))$, let $l(u)$ denote the maximum number of edges in a path in $E(u)$.

Claim 3.2. *For all* $u \in \mathrm{vert}(\tilde{Q}_n(0, n-1))$ *such that* $l(u) < n-1$, *there exists* $v \in \mathrm{vert}(\tilde{Q}_n(0, n-1))$ *adjacent to* u *with* $l(v) \geqslant \min\{n-1, 5/4l(u)\}$.

If $l(u) \leqslant 4$, then we let $v = u + x^e$ for some $e \notin E(u)$ incident to one endnode of a path of length $l(u)$ in $E(u)$, and such that $v \in \mathrm{vert}(\tilde{Q}_n(0, n-1))$. Therefore, v is adjacent to u and $l(v) \geqslant \min\{n-1, 5/4l(u)\}$. If $E(u) = n-2$, then we let $v = u + x^e$ for some $e \notin E(u)$ incident to one endnode of each of the two connected components in $E(u)$ (one is possibly an isolated node). Therefore, Hamiltonian cycle v is adjacent to u and $l(v) = n-1$.

Thus, the claim holds whenever $l(u) \leqslant 4$ or $E(u) = n-2$, and we only need to show that if $l(u) > 4$ and $E(u) < n-2$, then there exists $v \in \mathrm{vert}(\tilde{Q}_n(0, n-1))$ adjacent to $u \in \mathrm{vert}(\tilde{Q}_n(0, n-1))$ with $l(v) \geqslant \min\{n-1, 5/4l(u)\}$. Let $u \in Q_n(h)$, so $E(u)$ is comprised of $n-h$ connected components. Since $E(u) < n-2$, we have $n-h \geqslant 3$. Let $s = \min\{n-h-3, \lceil (l(u)-3)/4 \rceil\}$, so $s \geqslant 0$, and write $l(u) = 4s + r + 2$ with $r \geqslant 1$.

We use the two 'gadgets' shown in Fig. 1. In gadget 2, $(\psi, \ldots, \omega)$ may consist of a single edge or a path from $E(u)$. Choose a path P_0 with length $l(u)$ in $E(u)$. Define s copies of gadget 1 (with elements subscribed by $j = 1, \ldots, s$), and one copy of gadget 2 (subscripted by $s+1$, except for nodes ψ and ω which remain unsubscripted), as follows. Let α_1 be an endnode of P_0. Identify γ_j with α_{j+1} for $j = 1, \ldots, s$. Let all edges ρ_j^i and σ_j^i be in P_0. Let path $(\psi, \ldots, \omega)$ consist of the last r edges of P_0. Finally, choose $s+2$ other paths $P_1, \ldots, P_{s+2}$ (each possibly consisting of an isolated node) and let β_1 be an endnode of P_1, and δ_{j-1} and β_j be the endnodes of P_j for $j = 2, \ldots, s+1$ (so $\delta_{j-1} = \beta_j$ if P_j is an isolated node), and δ_{s+2} be an endnode of P_{s+2}.

From this construction, we note that the edge set $\{\rho_j^i\colon i = 1, \ldots, s; j = 1, 2\} \cup \{\sigma_j^i\colon i = 1, \ldots, s; j = 1, 2\} \cup \{\rho_{s+1}^1, \sigma_{s+1}^1\} \cup E(\psi, \ldots, \omega) = E(P_0)$. Define v by replacing in $E(u)$ all the σ_j^i edges with all the τ_j^i edges. Note that $v \in \mathrm{vert}(\tilde{Q}_n(h+s+2))$ and $l(v) \geqslant l(u)+s+2$. If $s = n-h-3$ then $E(u) = \bigcup_{i=0}^{s+2} E(P_i)$ and $l(v) = n-1$. Otherwise, $s = \lceil (l(u)-3)/4 \rceil$ and $l(u) = 4s + r + 2$ with $1 \leqslant r \leqslant 4$. Therefore, $l(v) \geqslant 5s + r + 4 \geqslant (5/4)l(u)$. Thus, to complete the proof of Claim 3.2 we only need to prove that u and v are adjacent on $\tilde{Q}_n(0, n-1)$. Assume, to the contrary, that they are not adjacent. Since $E(u) \cap E(v) = \{\rho_j^i\colon i = 1, \ldots, s; j = 1, 2\} \cup \{\rho_{s+1}^1\} \cup E(\psi, \ldots, \omega) \cup \bigcup_{i=1}^{s+2} E(P_i)$, the following pairs of edges are NC w.r.t. u, v:

(i) σ_j^1 and τ_j^i for all $j = 1, \ldots, s+1$ and $i = 1, 2, 3$;

(ii) σ_j^2 and τ_j^2 for all $j=1,\dots,s$; and
(iii) σ_j^2 and τ_{j+1}^2 for all $j=1,\dots,s$.
Let $\{y_1,\dots,y_T\} \in \mathrm{vert}(\tilde{Q}_n(0,n)) \setminus \{u,v\}$ be a nonadjacency-certificate for u,v, and consider any y_k in this set. If $\sigma_1^1 \in E(y_k)$ then we claim that $\sigma_j^1 \in E(y_k)$ for all $j=1,\dots,s+1$. Indeed, otherwise let $q\geqslant 2$ be the smallest index such that $\sigma_q^1 \notin E(y_k)$; then $\sigma_{q-1}^1 \in E(y_k)$ and we may apply Claim 3.1: by (i) above, $\tau_{q-1}^i \notin E(y_k)$ for all $i=1,2,3$, and then, by (ii), $\sigma_{q-1}^2 \in E(y_k)$; but then, by (iii), $\tau_q^2 \notin E(y_k)$ and, by (i) $\sigma_q^1 \in E(y_k)$, a contradiction. Thus, all $\sigma_j^1 \in E(y_k)$. As just seen, Claim 3.1 now implies that all $\tau_j^i \notin E(y_k)$ and therefore all $\sigma_j^2 \in E(y_k)$. Thus, we must have $y_k=u$, a contradiction with the fact that y_k is part of a nonadjacency-certificate for u,v.

Thus, we must have $\sigma_1^1 \notin E(y_k)$. By Claim 3.1 and (i) above this implies that $\tau_1^2 \in E(y_k)$. We claim that $\tau_j^2 \in E(y_k)$ for all $j=1,\dots,s+1$. Indeed, otherwise let $q\geqslant 2$ be the smallest index such that $\tau_q^2 \notin E(y_k)$; then $\tau_{q-1}^2 \in E(y_k)$ and we apply again Claim 3.1: by (ii) above, $\sigma_{q-1}^2 \notin E(y_k)$, and then, by (iii), $\tau_q^2 \in E(y_k)$, a contradiction. Thus, all $\tau_j^2 \in E(y_k)$. Claim 3.1 now implies that all $\sigma_j^i \notin E(y_k)$ and therefore all $\tau_j^i \in E(y_k)$. Thus, we must have $y_k=v$, again a contradiction with the fact that y_k is a part of a nonadjacency-certificate for u,v. Thus, there exists no nonadjacency-certificate for u,v, and u and v must therefore be adjacent. This completes the proof of Claim 3.2. □

Using Claim 3.2 repeatedly, a path can be constructed from any vertex in $\tilde{Q}_n(0,n-1)$ to the Hamiltonian Path Polytope. Upper bounds for the length of these paths are given in Claim 3.3.

Let $d(S,T)$ denote the distance on $\tilde{Q}_n(0,n-1)$ between two set of vertices S,T. Let 0^t denote the all-zero vector.

Claim 3.3. (a) *For $l=1,\dots,n-1$ it holds that $d(0^t,\mathrm{vert}(Q_n(l)))\leqslant\lceil {}^{(5/4)}\log l\rceil+1$.*

(b) *For $\alpha=1,\dots,n-2$ and $l=\alpha+1,\dots,n-1$, and for any $u\in\mathrm{vert}(Q_n(\alpha))$ it holds that $d(u,\mathrm{vert}(Q_n(l)))\leqslant\lceil {}^{(5/4)}\log l/\alpha\rceil$.*

Claim 3.3 directly results from Claim 3.2 and the fact that the all-zero vector is adjacent to any vertex of $Q_n(1)$.

Now, take any pair of vertices u,v in $\tilde{Q}_n(0,n-1)$. There are (Claim 3.3) Hamiltonian cycles u',v' such that both $d(u,u')$ and $d(v,v')$ are at most $\lceil {}^{(5/4)}\log(n-1)\rceil+1$. Moreover (Claim 3.3), there are paths from the all-zero vector to both u' and v' of length at most $\lceil {}^{(5/4)}\log(n-1)\rceil+1$. Hence, $d(u,v)\leqslant 4\lceil {}^{(5/4)}\log(n-1)\rceil+4$, so that the diameter of $\tilde{Q}_n(0,n-1)$ is at most $4\lceil {}^{5/4}\log(n-1)\rceil+4$. In a similar way, a path between any two vertices on $\tilde{Q}_n(0,n)$ of length at most $4\lceil {}^{(5/4)}\log n\rceil+4$ can be constructed. The reader may check this fact, while taking into account that two vertices are adjacent on $\tilde{Q}_n(0,n)$ if they are adjacent on $\tilde{Q}_n(0,n-1)$. □

Conjectures. (a) *The diameter of the skeleton of $\tilde{Q}_n(0,n)$ is at most $2\lceil {}^{(5/4)}\log n\rceil+4$.*

(b) *For* $\alpha = 1,\ldots,n$ *it holds that the diameter of the skeleton of* $\tilde{Q}_n(\alpha,n)$ *is at most* $2\lceil {}^{(5/4)}\log n/\alpha \rceil + 2$.

Note that if the Grötschel and Padberg (GP) conjecture that the diameter of $Q_n(n)$ equals two holds, then both conjectures are true. This can be easily checked with Claim 3.3. In the following theorem, we give lower bounds for the diameters of the partial monotonizations.

Theorem 3.2. *Let* $1 \leqslant k \leqslant n$ *and* $0 \leqslant \alpha \leqslant k$.
(a) $\operatorname{diam}\operatorname{skel}(\tilde{Q}_n(0,n)) \geqslant 1 + \lceil {}^3\log k \rceil$.
(b) $\operatorname{diam}\operatorname{skel}(\tilde{Q}_n(\alpha,k)) \geqslant \lceil {}^3\log k/\alpha \rceil$.

Proof. First of all, note that the all-zero vector is only adjacent to vertices of $Q_n(1)$. Hence, it is sufficient to show that for any two adjacent vertices $u \in Q_n(k)$ and $v \in Q_n(k')$, $1 \leqslant k < k' \leqslant n$, it holds that $k' < 3k$. Take any vertex u of $Q_n(k)$ (i.e. $E(u)$ is a collection of γ paths in K_n), and vertex v of $Q_n(k')$ adjacent to u. If $k' = k+1$ then $k' \leqslant 2k$ for all $k \geqslant 1$. Hence, without loss of generality, assume that $k' - k \geqslant 2$. Moreover, take any $e \in E(v) \setminus E(u)$. Note that $(E(u) \cup e) \notin \mathscr{A}_n(0) \cup \cdots \cup \mathscr{A}_n(n)$, because u and v are adjacent and $|k'-k| \geqslant 2$. Hence, an edge in $E(v) \setminus E(u)$ is either: (i) an edge that closes path j in $E(u)$ to a cycle ($j = 1,\ldots,p$), or (ii) an edge that causes a claw in $E(u)$. The number of edges in (i) is at most γ. The number of edges in (ii) is at most $2(k-\gamma)$, since there are at most $k-\gamma$ intermediate nodes of paths in $E(u)$. Hence, $|E(v)| = |E(u) \cap E(v)| + |E(v) \setminus E(u)| \leqslant k + \gamma + 2(k-\gamma) = 3k - \gamma$, so that $k' < 3k$. $\square$

As a consequence of Theorems 3.1 and 3.2, we have that the diameters of the Hamiltonian Cycle Polytope and the Hamiltonian Path Polytope are $\theta(\log n)$.

4. The number of 0-faces of level polytopes and partial monotonizations

The number of vertices of a polytope P can be seen as an indicator of its complexity. It yields an upper bound for the number of nondegenerate steps when applying simplex-like methods for finding optimal solutions on it. We derive an analytic, as well as a generating function expression for the number of 0-faces of the level polytopes. The number of 0-faces of partial monotonizations of these level polytopes can be determined by adding the number of 0-faces of the relevant level polytopes. Let $I_m(x)$ be the indicator function, defined by $I_m(x) = 1$ if $x = m$ and $I_m(x) = 0$ if $x \neq m$. Let $(i_1,\ldots,i_n)$ be a n-tuple of positive integers, and let $k = \max\{i_1,\ldots,i_n\}$. By $N(i_1,\ldots,i_n)$ is denoted the number of permutations that keep $(i_1,\ldots,i_n)$ unchanged. One can easily check that

$$N(i_1,\ldots,i_n) = \prod_{m=1}^{k} \left(\sum_{j=1}^{n} I_m(i_j) \right)!.$$

Theorem 4.1. *For* $k=0,\dots,n-1$ *the following holds*:

$$|\mathscr{A}_n(k)| = \sum_{\tau=1}^{\min\{k,n-k\}} \sum_{i_1\geqslant\cdots\geqslant i_\tau; i_1+\cdots+i_\tau=k} \frac{n!}{2^\tau (n-\tau-k)! N(i_1,\dots,i_\tau)}$$

$$= \frac{1}{k!}\left.\frac{\mathrm{d}^{n+k}}{\mathrm{d}x^n\,\mathrm{d}y^k}\right|_{x=y=0} \exp\left(\frac{x(2-xy)}{2(1-x)}\right).$$

Proof. The first expression follows from routine counting arguments. The second one is a generating function expression. Both are derived in the appendix. □

Using Theorem 4.1, brute force calculations show that the number of 0-faces of $Q_n(k)$, $k=0,\dots,n$, has a unimodal structure for $n\leqslant 10$. This leads us to the following conjecture.

Conjecture. *The number of* 0-*faces of* $Q_n(k)$ *has a unimodal structure with respect to* k $(0\leqslant k\leqslant n)$.

5. The Hamiltonian path–cycle polytope

In Section 2 it was shown that the Hamiltonian Path–Cycle Polytope (HPC-polytope) is the smallest full-dimensional partial monotonization of the Hamiltonian Cycle Polytope. There are several reasons to use the HPC-polytope instead of the full monotonization $\tilde{Q}_n(0,n)$ of the Hamiltonian Cycle Polytope. First of all, if the Grötschel Padberg (GP) conjecture stating that the diameter of the Hamiltonian Cycle Polytope equals two holds, then the diameter of $\tilde{Q}_n(0,n)$ is much larger than that of $Q_n(n)$; Corollary 3.3 states that the diameter of the Mon HC-polytope is $\theta(\log n)$, while the diameter of the HPC-polytope is conjectured to be at most four. Secondly, the number of 0-faces of $\tilde{Q}_n(n-1,n)$ is much smaller than that of $\tilde{Q}_n(0,n)$, although this does not automatically imply that the HPC-polytope is preferable from a computational (cutting planes or branch-and-cut) point of view. Furthermore, the HPC-polytope is Hamiltonian connected and its connectivity is $\frac{1}{2}(n)(n-1)$; see [7,11]. As a consequence of Theorem 2.2, we have that the HC-polytope is a face of the HPC-polytope and that the HP-polytope is a facet of the HPC-polytope.

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Appendix Proof of Theorem 4.1

Let $k \in \{0,\dots,n\}$, and let v be any vertex in the kth level. A path-collection $\{p_1,\dots,p_\tau\}$ in a graph G is a set of τ edge-disjoint paths in G; for $\tau=0$, $\{p_1,\dots,p_\tau\}$ is defined as $\emptyset$. Let $E(v)$ be a path-collection of τ paths in K_n. Clearly, $1 \leqslant \tau \leqslant \min\{k, n-k\}$, since the number of paths cannot exceed the number of edges in v, and since $\tau + k$ nodes are needed to obtain τ paths out of k edges. Let i_j be the length of path j $(j = 1,\dots,T)$. Let $i_1 \geqslant \cdots \geqslant i_\tau$. Choose $n + k$ nodes in K_n, and choose a permutation $P = (v_1,\dots,v_{n+k})$ of these nodes. There are $\binom{n}{\tau+k}(n+k)! = n!/(n-\tau-k)!$ ways for choosing P. Obtain a path-collection $(p_1,\dots,p_\tau)$ from P, by letting path j be $v_{(i_1+1)+\cdots+(i_{j-1}+1)+1},\dots,v_{(i_1+1)+(i_j+1)}$ for all j. Reversing the nodes in any path j or exchanging paths with equal length does not change the path-collection. Therefore, there are $2^\tau N(i_1,\dots,i_\tau)$ permutations that lead to the same path-collection.

Define $\delta_m^k(p)$ as the number of path-collections in K_m, such that all paths have length at most p and the sum of the length of all paths in the path-collection is k.

Claim A.1. *For* $p = 0,\dots,m-1$ *and* $k = 0,\dots,n-1$ *it holds that* $\delta_m^k(p) = 0$ *iff* $mp/(p+1) < k$.

(i) *Clearly, the number of edges satisfies* $k \leqslant p\tau$, *and the number of nodes satisfies* $\tau + k \leqslant m$. *Hence*, $k \leqslant mp/(p+1)$.

(ii) *Let* $k \leqslant mp/(p+1)$. *If* m *is a multiple of* $p+1$, *choose* $m/(p+1)$ *paths of length* p *and then leave out* $mp/(p+1) - k$ *edges. So, without loss of generality, assume that* m *is not a multiple of* $p+1$. *Let* $m=m'+m''$, *where* m' *is a multiple of* $p+1$ *and* $1 \leqslant m'' \leqslant p$. *Note that in the inequality* $k \leqslant pm/(p+1) = pm'/(p+1) + m'' - m''/(p+1)$, *only the last term does not have an integer value, so* $k \leqslant pm/(p+1) + m'' - 1$. *Choose* $m'/(p+1)$ *paths of length* p *and one path through the remaining* m'' *nodes. The total length is* $m'p/(p+1) + m'' - 1 \geqslant k$.

From (i) and (ii) we conclude that a path-collection consisting of k edges in K_m exists iff $k \leqslant mp/(p+1)$.

Define $\lambda_n(i,p)$ as the number of path-collections consisting of i paths of length p in K_n.

Claim A.2. *Let* $n \geqslant 3$, $p \geqslant 1$ *and* $0 \leqslant i \leqslant \lfloor n/(p+1) \rfloor$. *Then*,

$$\lambda_n(i,p) = \frac{n!}{2^i i!(n - i(p+1))!}.$$

We have to calculate the number of path-collections $\{p_1,\dots,p_i\}$ in which each path has length p. Path p_1 can be chosen in $n(n-1)\dots(n-p)/2$ ways. There are $(n-p-1)\dots(n-2p-1)/2$ possibilities for choosing path p_2 through the remaining nodes, and so on. The product of the i possibility factors for choosing the paths $p_1,\dots,p_i$ is $n!/[2^i(n-i(p+1))!]$. Because a permutation of a path-collection does not alter the path-collection, the last formula has to be divided by i!.

Claim A.3. *For $n\geqslant 0$, $p=0,\dots,m-1$ and $p=0,\dots,n-1$ it holds that*

$$\delta_m^k(p)=\begin{cases}\sum_{i=0}^{\lfloor k/p\rfloor}\lambda_m(i,p)\delta_{m-i(p+1)}^{k-ip}(p-1) & \text{for } \frac{k}{p}\leqslant\frac{m}{p+1},\\ 0 & \text{for } \frac{k}{p}>\frac{m}{p+1},\\ 1 & \text{for } k=0,\ p=0.\end{cases}\tag{1}$$

Let $k/p\leqslant m/(p+1)$. Choose i paths of length p in K_m ($\lambda_m(i,p)$ possibilities). The remaining paths, containing $k-ip$ edges, have to be constructed using the remaining $m-i(p+1)$ nodes. The second part follows from Claim A.1. $\delta_m^0(0)=0$ for all m, since there is one way to choose a path of length zero.

Finally, we derive the generating function expression for the number of 0-faces of $Q_n(p)$. To that end, define

$$H_p(x,y)=\sum_{m,k=0}^{\infty}\delta_m^k(p)\frac{x^m}{m!}y^k.$$

From the recursive formula for $\delta_m^k(p)$ in Claim A.3 we have, multiplying the terms on both sides of the equality sign with $x^m/m!$ and summing the result from $m=0$ to ∞:

$$\begin{aligned}
H_p(x,y)&=\sum_{m,k;k/p\leqslant m/(p+1)}\ \sum_{i=0}^{\lfloor k/p\rfloor}\lambda_m(i,p)[\delta_{m-i(p+1)}^{k-ip}(p-1)]\frac{x^m}{m!}y^k\\
&=\sum_{m,k;k/p\leqslant m/(p+1)}\ \sum_{i=0}^{\lfloor k/p\rfloor}\frac{m!}{2^i i!(m-i(p+1))!}[\delta_{m-i(p+1)}^{k-ip}(p-1)]\frac{x^m}{m!}y^k\\
&=\sum_{m,k;k/p\leqslant m/(p+1)}\ \sum_{i=0}^{\lfloor k/p\rfloor}\frac{1}{2^i i!}x^{i(p+1)}y^{ip}[\delta_{m-i(p+1)}^{k-ip}(p-1)]\\
&\quad\times\frac{x^{m-i(p+1)}}{(m-i(p+1))!}y^{k-ip}\\
&=\sum_{i=0}^{\infty}\ \sum_{m,k;k/p\leqslant m/(p+1),k\geqslant ip}\frac{1}{2^i i!}x^{i(p+1)}y^{ip}[\delta_{m-i(p+1)}^{k-ip}(p-1)]\\
&\quad\times\frac{x^{m-i(p+1)}}{(m-i(p+1))!}y^{k-ip}\\
&=\sum_{i=0}^{\infty}\frac{1}{2^i i!}x^{i(p+1)}y^{ip}\sum_{m,k;(k-ip)/p\leqslant(m-i(p+1))/(p+1),k-ip\geqslant 0}[\delta_{m-i(p+1)}^{k-ip}(p-1)]\\
&\quad\times\frac{x^{m-i(p+1)}}{(m-i(p+1))!}y^{k-ip}\\
&=H_{p-1}(x,y)\sum_{i=0}^{\infty}\frac{1}{2^i i!}x^{i(p+1)}y^{ip}=H_{p-1}(x,y)\exp\left[\frac{x^{p+1}y^p}{2}\right]\\
&=\cdots=\exp\left[\frac{1}{2}(x^2y+x^3y^2+\cdots+x^{p+1}y^p)\right]\times H_0(x,y)
\end{aligned}$$

$$
\begin{aligned}
(*) &= \exp\left[\frac{1}{2}(x^2y + x^3y^2 + \cdots + x^{p+1}y^p) + x\right] \\
&= \exp\left[\frac{1}{2}x\left(\frac{1-(xy)^{p+1}}{1-xy}\right) + \frac{1}{2}x\right] = \exp\left[\frac{x(2-xy-(xy)^{p+1})}{2(1-xy)}\right],
\end{aligned}
$$

where we used

$$
(*)\left\{ H_0(x,y) = \sum_{m,k=0}^{\infty} \delta_m^k(0)\frac{x^m}{m!}\, y^k = \sum_{m=0}^{\infty} \frac{x^m}{m!} = \mathrm{e}^x. \right.
$$

Hence, the number of vertices in level p is

$$
\begin{aligned}
|\mathscr{A}_n(k)| = \delta_n^k(k) &= \frac{n!}{n!k!}\left.\frac{\mathrm{d}^{n+k}}{\mathrm{d}x^n\,\mathrm{d}y^k}\right|_{x=y=0} H_k(x,y) \\
&= \frac{1}{k!}\left.\frac{\mathrm{d}^{n+k}}{\mathrm{d}x^n\,\mathrm{d}y^k}\right|_{x=y=0} \exp\left[\frac{x(2-xy-(xy)^{k+1})}{2(1-xy)}\right] \\
&= \frac{1}{k!}\left.\frac{\mathrm{d}^{n+k}}{\mathrm{d}x^n\,\mathrm{d}y^k}\right|_{x=y=0} \exp\left[\frac{x(2-xy)}{2(1-xy)}\right]. \qquad \square
\end{aligned}
$$

6. For further reading

The following references are also of interest to the reader: [2,3,6,8,9]

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